Review on Week 8

Continuous Function

The concept of a continuous function is very important in analysis. We always encounter continuous functions since we learn Mathematics. Almost every elementary functions are continuous. In fact, continuity of a function is crucial for us to "draw" its graph.

Recall the definition of the limit of a function:

Definition. Let c be a cluster point of $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. The limit of f at c is L, denoted $L = \lim_{x \to c} f(x)$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x - c| < \delta$ and $x \in A$,

$$|f(x) - L| < \varepsilon.$$

Definition (c.f. Definition 5.1.1)**.** Let $c \in A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. f is said to be *continuous* at c if for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x - c| < \delta$ and $x \in A$,

$$|f(x) - f(x)| < \varepsilon.$$

Also, f is said to be *continuous on* A if f is continuous at every $c \in A$.

Remark. The definition of limit and continuity is very similar, but be careful about the requirement of the point c.

- For limit, we **require** c to be a cluster point of A but c **need not** lies in A.
- For continuity, we **require** c to lie in A but c **need not** be a cluster point of A.

Hence if $c \in A$ and c is a cluster point in A. Then f is continuous at c if and only if

$$f(c) = \lim_{x \to x} f(x).$$

Let's look at the following special example:

Example. Consider $A = \mathbb{N}$. Then any functions $f : \mathbb{N} \to \mathbb{R}$ is continuous.

Proof. Let $c \in \mathbb{N}$ and $\varepsilon > 0$. Take $\delta = 1$. Then $|x - c| < \delta = 1$ and $x \in \mathbb{N}$ implies that $x = c$. Hence

$$|f(x) - f(c)| = 0 < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that f is continuous at c. Since $c \in \mathbb{N}$ is arbitrary, it follows that f is continuous on $\mathbb{N}$. □

Examples

Example 1 (c.f. Example 5.1.6(a)-(e))**.** The following are all continuous functions on their **maximum domains of definition**:

(a) The constant function $f_1(x) = b$.

(b) The identity map $f_2(x) = x$.

(c) The square function $f_3(x) = x^2$.

(d) The reciprocal function $f_4(x) = 1/x$.

(e) The absolute value function $f_5(x) = |x|$.

Let show that f_4 is continuous on $\mathbb{R} \setminus \{0\}$. Note that for any $x, c \neq 0$,

$$|f_4(x) - f_4(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| = \frac{1}{|x||c|}|x - c|.$$

Let $c \neq 0$ and divide into two cases:

- $c > 0$: First note that if $|x - c| < \frac{1}{2}c$, then $0 < \frac{1}{2}c < x < \frac{3}{2}c$. Hence

  $$\frac{1}{|x||c|}|x - c| = \frac{1}{cx}|x - c| < \frac{2}{c^2}|x - c|, \quad \text{whenever } |x - c| < \frac{1}{2}c \text{ and } x \neq 0.$$

  Let $\varepsilon > 0$. Take $\delta = \min\{\frac{1}{2}c, \frac{c^2}{2}\varepsilon\}$. Then whenever $|x - c| < \delta$,

  $$|f_4(x) - f_4(c)| = \frac{1}{|x||c|}|x - c| < \frac{2}{c^2}|x - c| < \frac{2}{c^2}\delta \leq \varepsilon.$$

- $c < 0$: Similar to the above case but take $\delta = \min\{-\frac{1}{2}c, \frac{c^2}{2}\varepsilon\}$. I leave the details.

Example 2 (c.f. Example 5.1.8(b))**.** Consider the function $g(x) = x \sin(1/x)$. It has limit 0 at $x = 0$ and hence we can define $G : \mathbb{R} \to \mathbb{R}$ by

$$G(x) = \begin{cases} g(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Then G is continuous on $\mathbb{R}$ and is called the *continuous extension* of g.

Proof. Observe that we have

Squeeze Theorem (c.f. 4.2.7)**.** *Let c be a cluster point of $A \subseteq \mathbb{R}$ and $f, g, h : A \to \mathbb{R}$ be functions. Suppose that*

$$f(x) \leq g(x) \leq h(x), \quad \forall x \in A, x \neq c \quad \textit{and} \quad \lim_{x\to c} f(x) = \lim_{x\to c} h(x).$$

Then $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \lim_{x\to c} h(x)$.

Note that since $-1 \le \sin(1/x) \le 1$ for $x \neq 0$,

$$-|x| \le g(x) \le |x|, \quad \forall c \neq 0.$$

Also, $\lim_{x\to 0}(-|x|) = \lim_{x\to 0} |x| = 0$. It follows that $\lim_{x\to 0} g(x) = 0$. □

Example 3 (c.f. Example 5.1.6(g))**.** Consider the Dirichlet's function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

Then h is not continuous on every every point $c \in \mathbb{R}$.

Proof. Observe that we have

Sequential Criterion (c.f. 5.1.3 and 5.1.4)**.** *A function $f : A \to \mathbb{R}$ is continuous at $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.*

Let $c \in \mathbb{R}$ and divide into two cases:

- $c \in \mathbb{Q}$. By the density of $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$, there is a sequence (x_n) of irrational number that converges to c. Hence $h(x_n) = 0$ for all $n \in \mathbb{N}$. Therefore

$$\lim_{n\to\infty} f(x_n) = 1 \neq 0 = h(c).$$

  It follows that h is discontinuous at c.

- $c \notin \mathbb{Q}$. Then the same arguement holds but we interchange $\mathbb{Q}$ with $\mathbb{R} \setminus \mathbb{Q}$.

In any cases, h is discontinuous at c. □

Exercises

Question 1 (c.f. Section 5.1, Ex.3)**.** Let $a < b < c$. Suppose that f is continuous on $[a, b]$, g is continuous on $[b, c]$ and $f(b) = g(b)$. Define h on $[a, c]$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in [a, b] \\ g(x) & \text{if } x \in [b, c] \end{cases}.$$

Prove that h is continuous at b.

Solution. Let $\varepsilon > 0$. Since f is continuous at b, there exist $\delta_1 > 0$ such that

$$|f(x) - f(b)| < \varepsilon, \quad \text{whenever } |x - b| < \delta_1 \text{ and } x \in [a, b].$$

Since g is continuous at b, there exist $\delta_2 > 0$ such that

$$|g(x) - g(b)| < \varepsilon, \quad \text{whenever } |x - b| < \delta_2 \text{ and } x \in [b, c].$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Then whenever $|x - b| < \delta$ and $x \in [a, c]$, consider the two cases:

- $x \in [a, b]$: Then $|h(x) - h(b)| = |f(x) - f(b)| < \varepsilon$.
- $x \in [b, c]$: Then $|h(x) - h(b)| = |g(x) - g(b)| < \varepsilon$.

In any cases, $|h(x) - h(b)| < \varepsilon$. Hence h is continuous at b.

Question 2 (c.f. Section 5.1, Ex.7)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at c and let $f(c) > 0$. Show that there is a $\delta > 0$ such that $f(x) > 0$ whenever $x \in (c - \delta, c + \delta)$.

Solution. Take $\varepsilon = f(c)/2 > 0$. Since f is continuous at c, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon = \frac{f(c)}{2}, \quad \text{whenever } |x - c| < \delta.$$

Hence if $x \in (c - \delta, c + \delta)$, i.e., $|x - c| < \delta$,

$$0 < \frac{1}{2}f(c) < f(x) < \frac{3}{2}f(c).$$

In particular, $f(x) > 0$.

Question 3 (c.f. Section 5.1, Ex.11)**.** Let $K > 0$ and $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition

$$|f(x) - f(y)| \leq K|x - y|, \quad \forall x, y \in \mathbb{R}.$$

Show that f is continuous at every point $c \in \mathbb{R}$.

Solution. Let $c \in \mathbb{R}$ and $\varepsilon > 0$. Take $\delta = \varepsilon/K$. Then whenever $|x - c| < \delta$,

$$|f(x) - f(c)| \leq K|x - c| < K\delta = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, f is continuous at c. Since $c \in \mathbb{R}$ is arbitrary, f is continuous on $\mathbb{R}$.